# **Lecture 3. Classification of second order partial differential equations**

## **Introduction**

The class of partial differential equations is very large.

We would like to simplify the partial differential equations.

We change the independent variables such that the equation with new variables has as soon as easy form, which is called the canonic form.

These canonic forms of the equations are the basis of its classification.

Then we will analyze the canonic form only.

## **3.1. Subject of analysis**

The general mathematical physics problem are the second order partial differential equations.

We have equations with two independent variables for the easiest case.

The general form of second order partial differential equation with two independent variables is



However, we will consider the linear equation with respect to the high derivatives

 (3.1)

where *aij*= *aij*(*x*,*y*).

## **3.2. Change of the independent variables**

Determine new independent variable



We choose these variables such that the equation (3.1) with respect to these variables are as soon as easy. Then we can analyze this easier equation and find its solution *u* as a function of *ξ* and *η*. By the final step, we return to the solution of the equation (3.1) as a function of *x* and *y* by the inverse transformation



Let *v* be the solution of the considered equation in new variables. We have the following equalities



Find the derivatives





Now determine the second order derivatives







Put the results to the formula (3.1). We obtain the equation for the function *v*  
that is linear with respect to the second derivatives. We get the equation

 (3.2)

where



and the function  does not depend from the second derivatives.

What is the sense of these transformations?

We choose the new variable such that the equation in new variable will be as soon as easy.

We choose the new variables such that one at least of the coefficients of the equality (3.2) is equal to zero.

## **3.3. Characteristic equation**

*Definition.* The characteristic equation for the equation (3.1) is the ordinary differential equation

 (3.3)

*Lemma*. If  where *c* is an arbitrary constant, is the general solution of the equation (3.3), then the function *ϕ* satisfies the equation

 (3.4)

*Proof*. The equality  is the concrete relation between *x* and *y.* We can interpreted it as the algebraic equation with respect to *y.* Find its solution  Consider a point  Determine a constant  Now we consider the equality  Now we determine  This is a partial solution of the equation (3.3). Therefore, we get

 (3.5)

Using the equality , we find the differential



Then we find



Put it to the equality (3.5). We obtain

 (3.6)

Determine here *x=x*0. Then  Now we get



Thus, the equality (3.4) is true for the arbitrary point  This complete the proof of the lemma.

## **3.4. Classification of the equation (3.1).**

Return to the consideration of the equation (3.3), which can be transformed to the equality



This is the square equation with respect to the derivative. It has two solutions

 (3.7)

 (3.8)

The properties of these equations depends from the value of the sign of the value



which is called the ***discriminant***.

*Definition*. The equation (3.1) is called ***hyperbolic*** at the concrete point, if  ***parabolic***, if  and ***elliptic***, if 

## **3.5. Hyperbolic equation.**

If our equation is hyperbolic, then we have to different equations (3.7) and (3.8) with real value at the right-hand side. Let  and  are its general solutions. Determine  Using the previous lemma, we determine  Therefore, the equality (3.2) is transformed to

 (3.9)

where



The equality (3.9) is the ***canonic form of the hyperbolic equation***.

Sometimes, one uses other canonic form. Determine the new variables



Then we have



Find the derivatives



Then the equality (3.9) is transformed to

 (3.10)

where  This is the second ***canonic form of the hyperbolic equation***.

## **3.6. Parabolic equation.**

For the parabolic case, the equalities (3.7) and (3.8) are equal. Let  be a general solution of this equation. Then we choose  and the function *η* is arbitrary. Calculate



because of the lemma and the equality  Now determine



because of the previous equality. In this case the equality (3.2) is transformed to

 (3.11)

where



The equality (3.11) is called the ***canonic form of the parabolic equation***.

## **3.7. Elliptic equation.**

For the elliptic case, the values at the right-hand sides of the equality (3.7) and (3.8) are complex. Besides, it has the same real parts, and its imagine parts are differ in the sign only. Then there are, complex adjoint values. Therefore, if  is the general solution of the equation (3.7), which is the complex function, then  is the general solution of the equation (3.8), where  is the adjoint function to *ϕ*.

Determine  Now the equality (3.2) is transformed to (3.9) again, but with complex values. For returning to the equation with real parameters, we determine the new variables



In this situation we have



Find the value



Using the properties of the complex numbers, determine the equalities  with respect to the variables *α* and *β*. Thus, the equality (3.8) is transformed to

 (3.10)

where  This is the second ***canonic form of the elliptic equation***.

## **3.8. Classification of known equations.**

Consider the string vibrating equation . Determine *D=*1. Thus, this equation is hyperbolic.

Consider the heat equation . Determine *D=*0. Thus, this equation is parabolic.

Consider the Poisson equation . Determine *D=*-1. Thus, this equation is elliptic.

## **3.9. Reduction of the concrete hyperbolic equation to the canonic form.**

Consider the equation . We have  Determine *D=*1. Then we have two characteristic equations *y' =*3 and *y' =*1. The general solutions of these equations are *y-*3*x=c* and *y-x=c.* Choose the new variables *ξ* = *y-*3*x* and *η = y-x.* Using the known formulas, determine the coefficients



Now we have the equality  and 

## **3.10. Reduction of the concrete parabolic equation to the canonic form.**

Consider the equation . We have  Determine *D=*0. Then we have the characteristic equation *y' =*2. It has the general solution *y-*2*x=c.* Choose the new variable *ξ* = *y-*2*x* and *η* is arbitrary, for example *η=x.* Using the known formulas, determine the coefficients



Now we have the equality .

Здесь надо добавить младшую производную. Иначе получим ОДУ

## **3.11. Reduction of the concrete elliptic equation to the canonic form.**

Consider the equation . We have  Determine *D=*-1. Then we have the characteristic equations *y' =*2±i. These equations have the general solutions   
*y-*2*x*+i*x=c* and *y-*2*x*-i*x=c.* Choose the new variable *ξ* = *y-*2*x* and *η=x.* Using the known formulas, determine the coefficients



## **Conclusions**

* The partial differential equations can be simplified by the exchange of the independent variables.
* The canonic forms are results of the exchange of the independent variables.
* The canonic form of the equation is equivalent to its initial form, but it is easier.
* There exists three canonic forms of the second order partial differential equations with two independent variables.
* There exists three types of the second order partial differential equations with two independent variables, which depend of the corresponding canonic forms.
* There are hyperbolic, parabolic and elliptic equations.
* The vibrating string equation is hyperbolic.
* The heat equation is parabolic.
* The Poisson equation is elliptic.
* Any equation can be reduced to the canonic form
* Equation canonic form depends from the type of equation.
* To reduce the hyperbolic equation to canonical form, two general solutions of the characteristic equation are used as new variables.
* To reduce the parabolic equation to canonical form, the general solution of the characteristic equation is used as new variable; second variable can be arbitrary.
* To reduce the parabolic equation to canonical form, the real and imagine part of general solution of the characteristic equation are used as new variables.

## **Task**

In Lecture 3, we considered the classification of second order partial differential equations with two independent variables



It is necessary to determine the type of the given equation and to reduce it the canonic form using the transformation to the new variables



**Table of equations**

|  |  |
| --- | --- |
| variant | equation |
| 1 |  |
| 2 |  |
| 3 |  |
| 4 |  |
| 5 |  |
| 6 |  |
| 7 |  |
| 8 |  |

It is necessary perform the following steps:

1. Calculate the value of the discriminant *D.*
2. Using the sign of *D* determine the sets of the plane *xy*, where the equation has the concrete type.
3. For the hyperbolic case, write two characteristic equations.
4. Find its general solutions.
5. Write these general solutions in the form  and .
6. Determine the new variables  
7. Calculate the coefficients of the equation in the new variables by the given formulas.
8. Determine the canonic form of the given equation for the hyperbolic case.
9. For parabolic case, consider the unique characteristic equation, determine variable  by previous method with arbitrary variable *η*, and repeat the actions of hyperbolic case.
10. For elliptic case, consider the first characteristic equation with complex parameters, find its general solution, write it is the form , choose the functions *ξ* and *η* as new variables, and repeat the actions of hyperbolic case.